

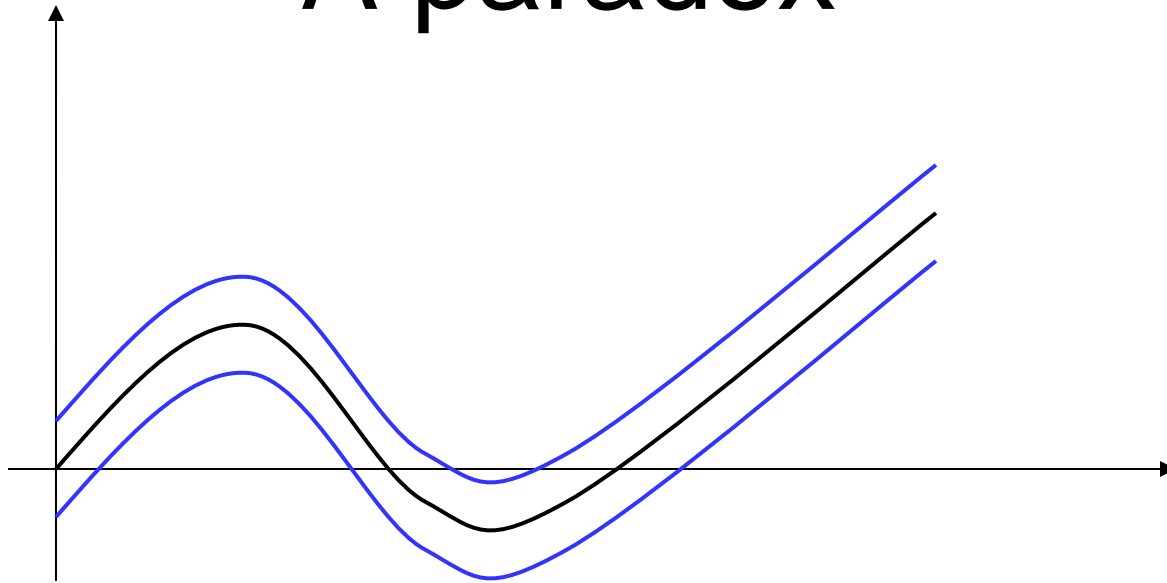


# **BROWNIAN MOTION**

## **A tutorial**

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# A paradox

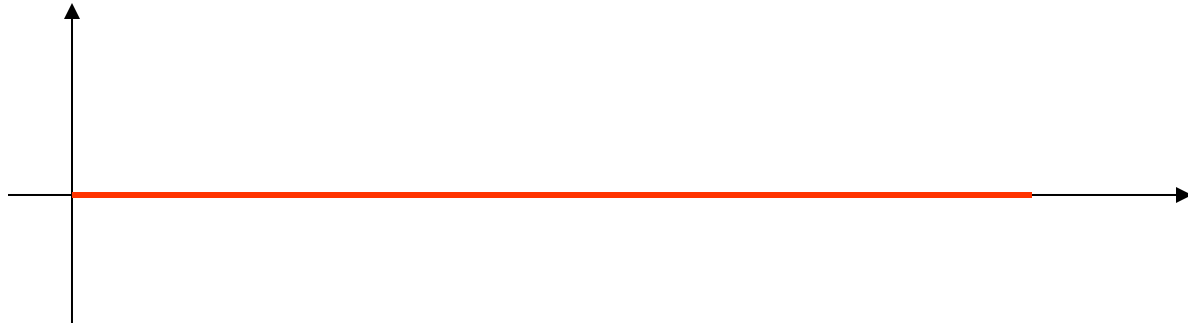


$$f : [0,1] \rightarrow \mathbb{R}, \quad \sup_{t \in [0,1]} |f''(t)| < \infty$$

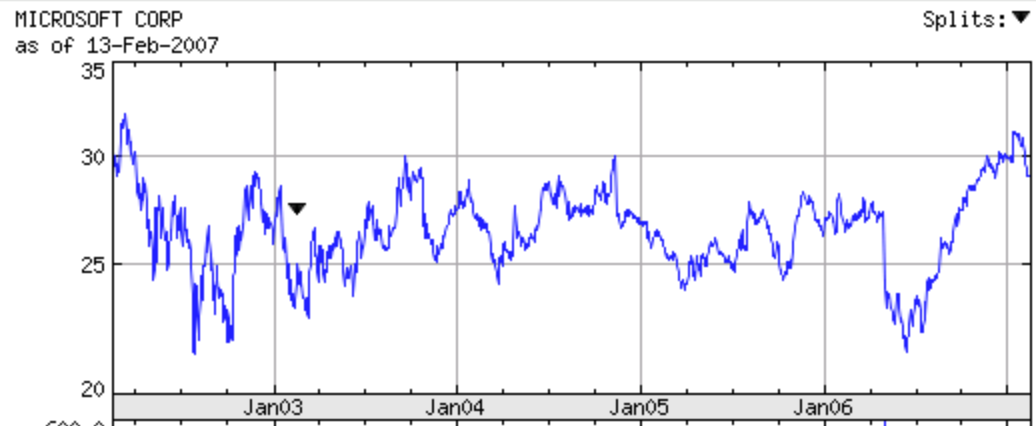
$$P(f(t) - \varepsilon < B_t < f(t) + \varepsilon, 0 < t < 1)$$

$$\approx c(\varepsilon) \exp\left(-\frac{1}{2} \int_0^1 (f'(t))^2 dt\right) \quad (*)$$

(\*) is maximized by  $f(t) = 0, t > 0$   
The most likely (?!?) shape of a  
Brownian path:



Microsoft stock  
- the last 5 years



# Definition of Brownian motion

Brownian motion is the unique process with the following properties:

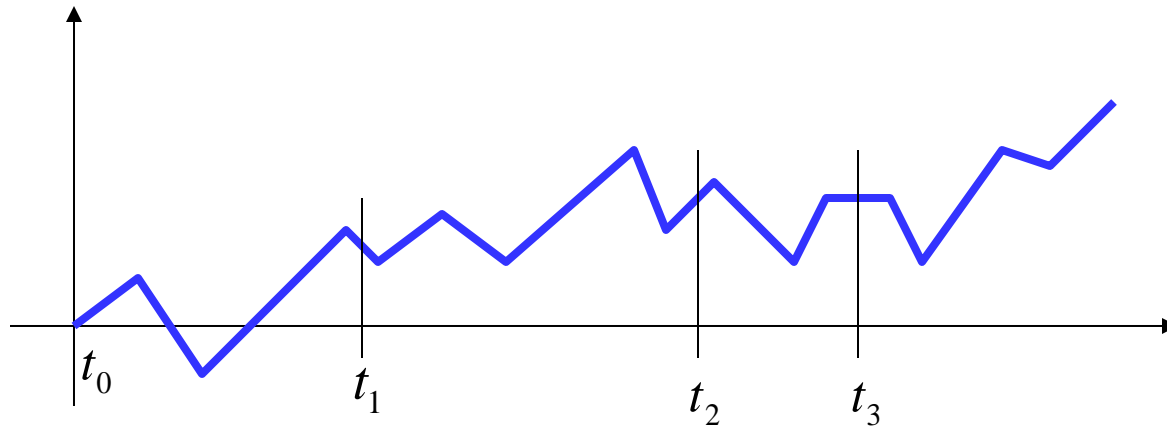
(i) No memory

(ii) Invariance

(iii) Continuity

(iv)  $B_0 = 0$ ,  $E(B_t) = 0$ ,  $Var(B_t) = t$

# Memoryless process



$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \dots$$

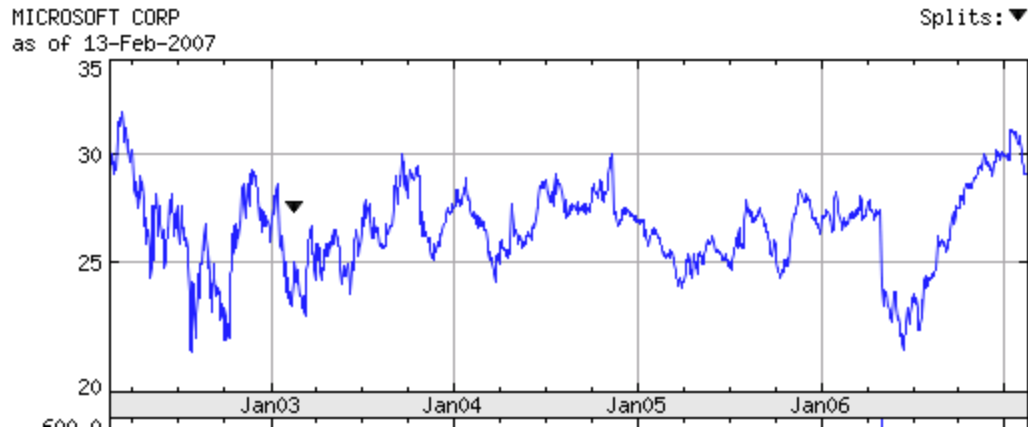
are independent

# Invariance

The distribution of  $B_{t+s} - B_s$   
depends only on  $t$ .

# Path regularity

- (i)  $t \rightarrow B_t$  is continuous a.s.
- (ii)  $t \rightarrow B_t$  is nowhere differentiable a.s.





# Why Brownian motion?

Brownian motion belongs to several families of well understood stochastic processes:

- (i) Markov processes
- (ii) Martingales
- (iii) Gaussian processes
- (iv) Levy processes



# Markov processes

$$\mathcal{L}\{B_t, t \geq s \mid B_s\} = \mathcal{L}\{B_t, t \geq s \mid B_u, 0 \leq u \leq s\}$$

The theory of Markov processes uses tools from several branches of analysis:

- (i) Functional analysis (transition semigroups)
- (ii) Potential theory (harmonic, Green functions)
- (iii) Spectral theory (eigenfunction expansion)
- (iv) PDE's (heat equation)

# Martingales

$$s < t \Rightarrow E(B_t | B_s) = B_s$$



Martingales are the only family of processes for which the theory of stochastic integrals is fully developed, successful and satisfactory.

$$\int_0^t X_s dB_s$$

# Gaussian processes

$B_{t_1}, B_{t_2}, \dots, B_{t_n}$  is multidimensional normal (Gaussian)

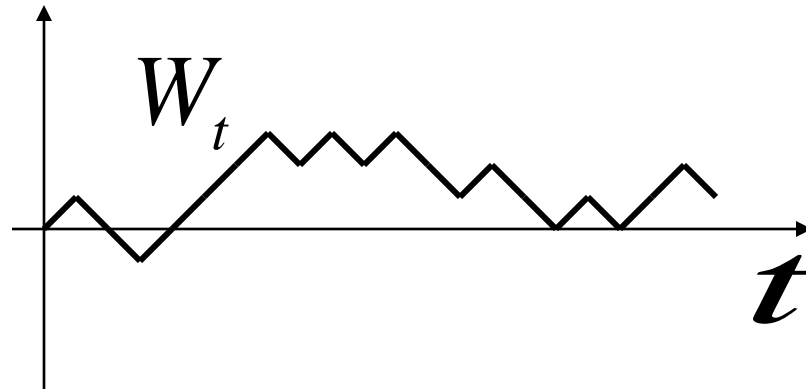
- (i) Excellent bounds for tails
- (ii) Second moment calculations
- (iii) Extensions to unordered parameter(s)

# The Ito formula

$$\int_0^t X_s dB_s = \lim_{n \rightarrow \infty} \sum_{k=0}^{nt} X_{k/n} (B_{(k+1)/n} - B_{k/n})$$

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds$$

# Random walk



Independent steps,  $P(\text{up})=P(\text{down})$

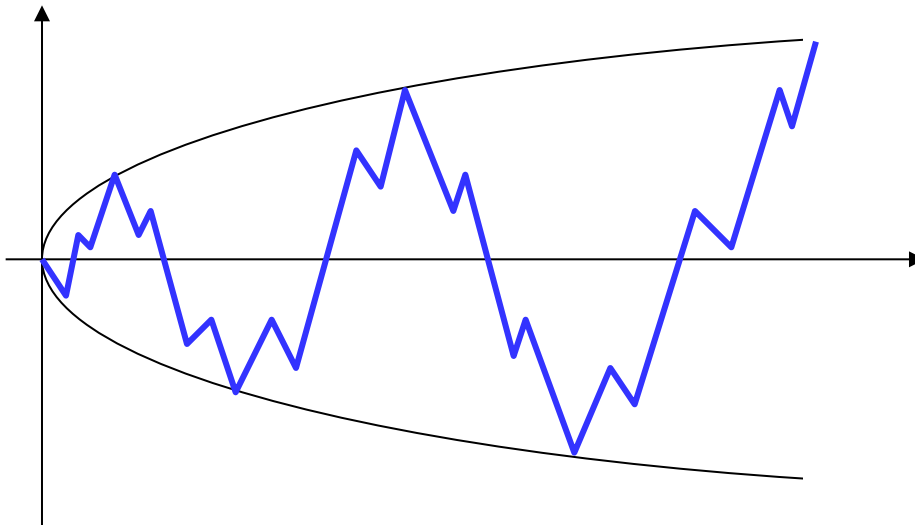
$$\left\{ \sqrt{a} W_{t/a}, t \geq 0 \right\} \xrightarrow{a \rightarrow \infty} \left\{ B_t, t \geq 0 \right\}$$

(in distribution)

# Scaling

Central Limit Theorem (CLT),  
parabolic PDE's

$$\{B_t, 0 \leq t \leq 1\} \stackrel{D}{=} \{\sqrt{a}B_{t/a}, 0 \leq t \leq 1\}$$



# Cameron-Martin-Girsanov formula

Multiply the probability of each Brownian path  $\{B_t, 0 \leq t \leq 1\}$  by

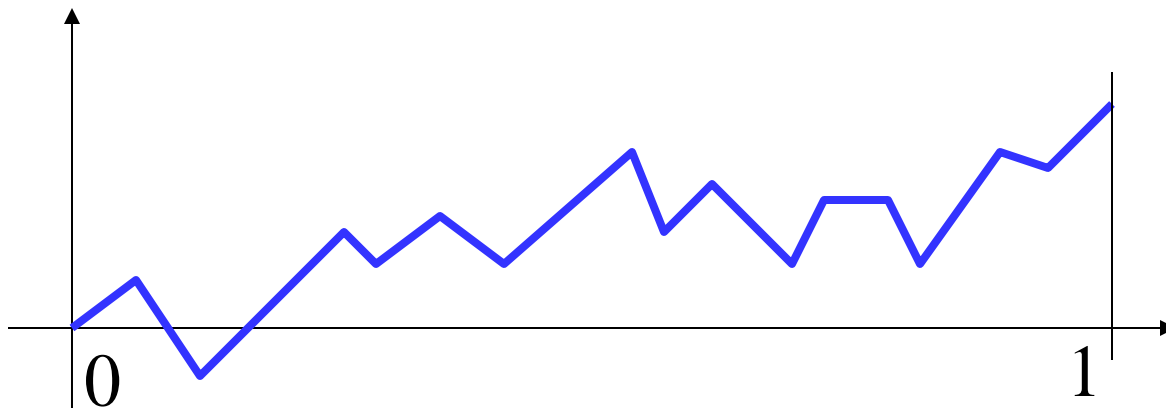
$$\exp\left(\int_0^1 f'(s)dB_s - \frac{1}{2}\int_0^1 (f'(s))^2 ds\right)$$

The effect is the same as replacing  $\{B_t, 0 \leq t \leq 1\}$  with  $\{B_t + f(t), 0 \leq t \leq 1\}$

# Invariance (2)

Time reversal

$$\{B_t, 0 \leq t \leq 1\} \stackrel{D}{=} \{B_{1-t} - B_1, 0 \leq t \leq 1\}$$





# Brownian motion and the heat equation

$u(x, t)$  – temperature at location  $x$  at time  $t$

Heat equation: 
$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{2} \Delta_x u(x, t)$$

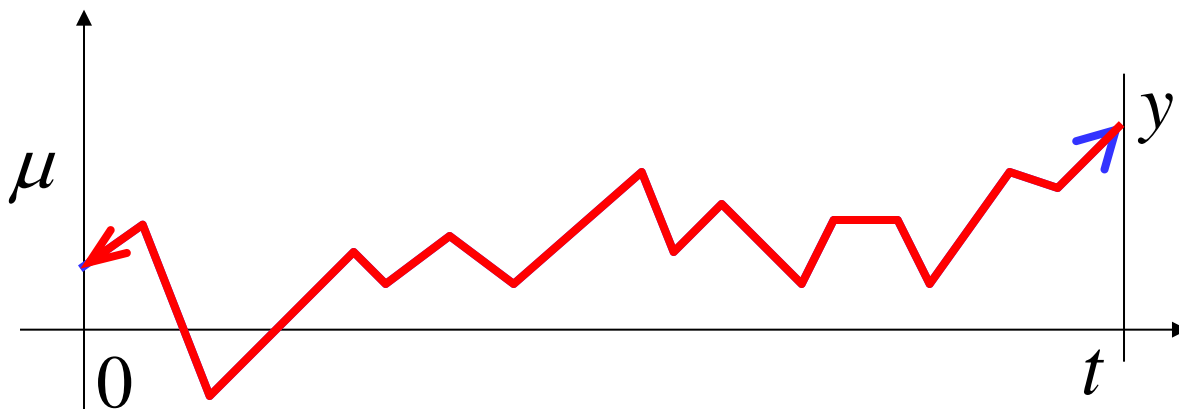
$$\mu(dx) = u(x, 0) dx$$

Forward  
representation

$$u(y, t) dy = P^\mu (B_t \in dy)$$

Backward representation  
(Feynman-Kac formula)

$$u(y, t) = Eu(B_t + y, 0)$$

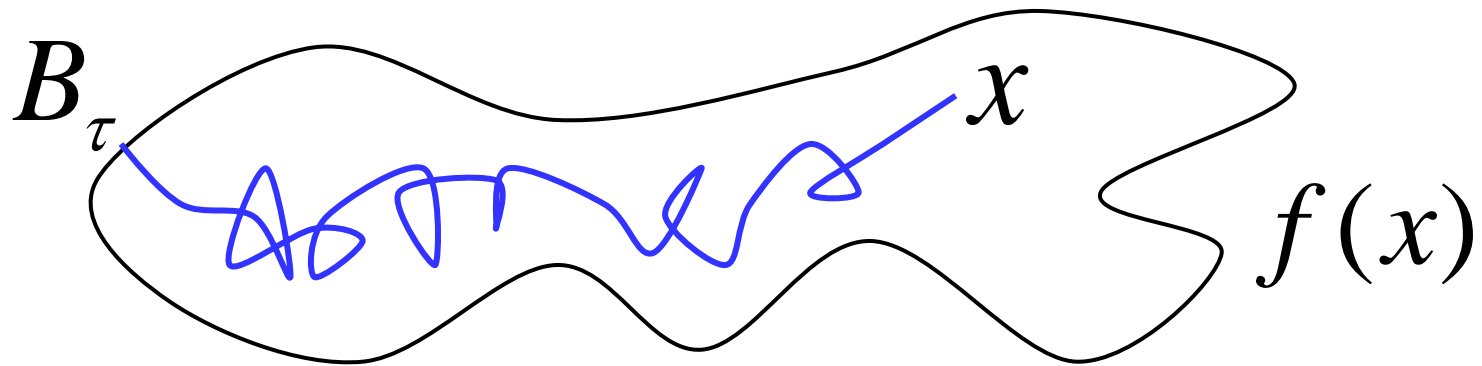


# Multidimensional Brownian motion

$B_t^1, B_t^2, B_t^3, \dots$  - independent 1-dimensional Brownian motions

$(B_t^1, B_t^2, \dots, B_t^d)$  - d-dimensional Brownian motion

# Feynman-Kac formula (2)



$$\frac{1}{2} \Delta u(x) - V(x)u(x) = 0$$

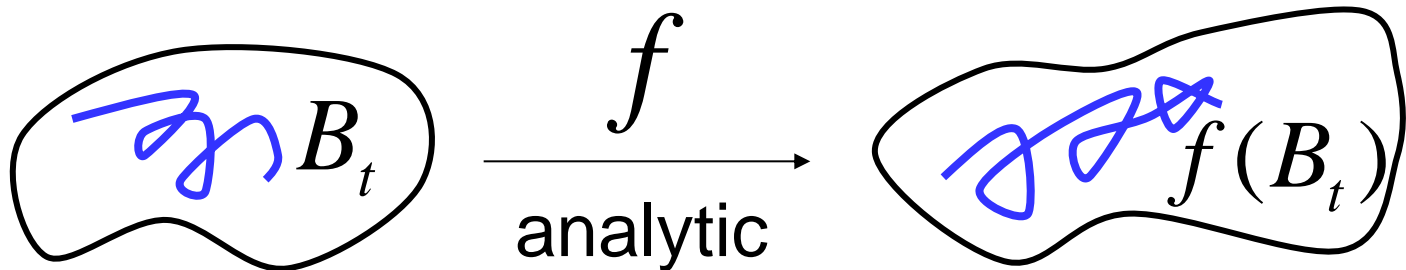
$$u(x) = E^x \left( f(B_\tau) \exp \left[ - \int_0^\tau V(B_s) ds \right] \right)$$

## Invariance (3)

The d-dimensional Brownian motion is invariant under isometries of the d-dimensional space. It also inherits invariance properties of the 1-dimensional Brownian motion.

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \exp(-x_1^2 / 2) \frac{1}{\sqrt{2\pi}} \exp(-x_2^2 / 2) \\ &= \frac{1}{2\pi} \exp(-(x_1^2 + x_2^2) / 2) \end{aligned}$$

# Conformal invariance



$$\{f(B_t) - f(B_0), t \geq 0\}$$

has the same distribution as

$$\{B_{c(t)}, t \geq 0\}, \quad c(t) = \int_0^t |f'(B_s)|^2 ds$$

# The Ito formula

## Disappearing terms (1)

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s + \frac{1}{2} \int_0^t \Delta f(B_s) ds$$

If  $\Delta f \equiv 0$  then

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s) dB_s$$

# Brownian martingales

Theorem (Martingale representation theorem).  
 $\{\text{Brownian martingales}\} = \{\text{stochastic integrals}\}$

$$M_t = \int_0^t X_s dB_s$$

$$E(M_t | M_s) = M_s, \quad M_t \in F_t^B = \sigma\{B_s, s \leq t\}$$

# The Ito formula

## Disappearing terms (2)

$$f(t, B_t) - f(t, B_0) = \int_0^t \frac{\partial}{\partial x} f(s, B_s) dB_s$$

$$+ \int_0^t \frac{\partial}{\partial s} f(s, B_s) ds + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(s, B_s) ds$$

$$Ef(t, B_t) - Ef(t, B_0)$$

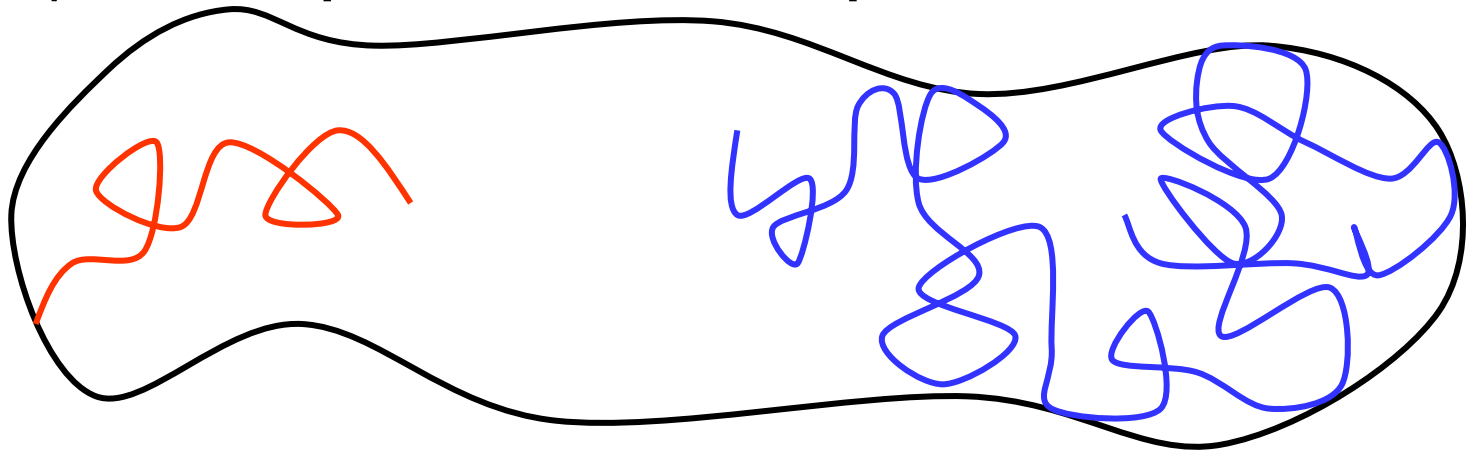
$$= E \int_0^t \frac{\partial}{\partial s} f(s, B_s) ds + \frac{1}{2} E \int_0^t \frac{\partial^2}{\partial x^2} f(s, B_s) ds$$



# Mild modifications of BM

Mild = the new process corresponds to the Laplacian

- (i) Killing – Dirichlet problem
- (ii) Reflection – Neumann problem
- (iii) Absorption – Robin problem



# Related models – diffusions

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt$$

- (i) Markov property – yes
- (ii) Martingale – only if  $\mu \equiv 0$
- (iii) Gaussian – no, but Gaussian tails

# Related models – stable processes

Brownian motion –  $dB = (dt)^{1/2}$   
Stable processes –  $dX = (dt)^{1/\alpha}$

- (i) Markov property – yes
- (ii) Martingale – yes and no
- (iii) Gaussian – no

Price to pay: jumps, heavy tails,  $0 < \alpha \leq 2$   
 $0 < 2 \leq 2$

# Related models – fractional BM

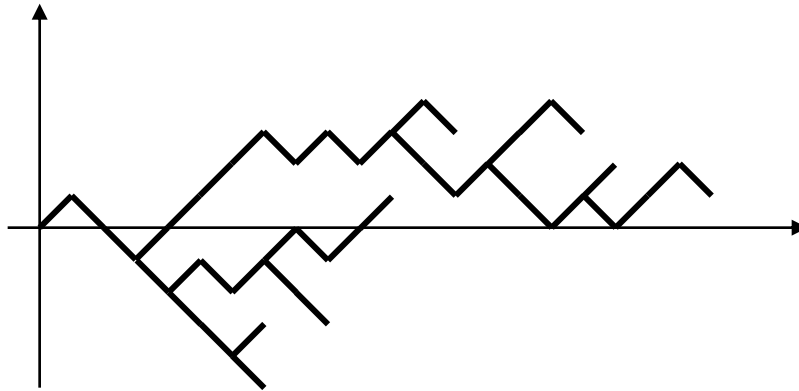
$$dX = (dt)^{1/\alpha}$$

- (i) Markov property – no
- (ii) Martingale – no
- (iii) Gaussian – yes
- (iv) Continuous

$$1 < \alpha < \infty$$

$$1 < 2 < \infty$$

# Related models – super BM



Super Brownian motion is related to

$$\Delta u = u^2$$

and to a stochastic PDE.



# Related models – SLE

Schramm-Loewner Evolution is a model for non-crossing conformally invariant 2-dimensional paths.

# Path properties

- (i)  $t \rightarrow B_t$  is continuous a.s.
- (ii)  $t \rightarrow B_t$  is nowhere differentiable a.s.
- (iii)  $t \rightarrow B_t$  is Holder  $(1/2 - \varepsilon)$
- (iv) Local Law if Iterated Logarithm

$$\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{2t \log |\log t|}} = 1$$

# Exceptional points

$$\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{2t \log |\log t|}} = 1$$

For any fixed  $s > 0$ , a.s.,

$$\limsup_{t \downarrow s} \frac{B_t - B_s}{\sqrt{2(t-s) \log |\log(t-s)|}} = 1$$

There exist  $s > 0$ , a.s., such that

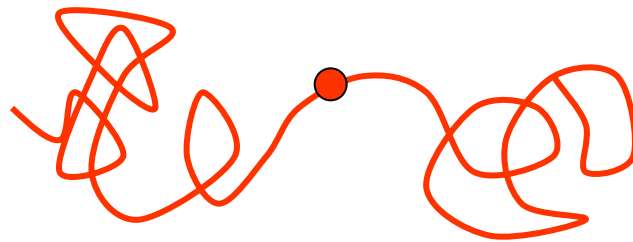
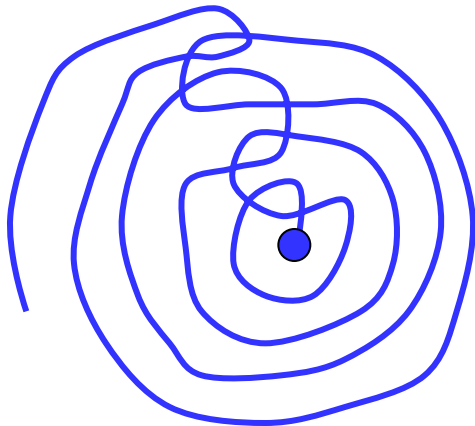
$$\limsup_{t \downarrow s} \frac{B_t - B_s}{\sqrt{2(t-s)}} \in (0, \infty)$$



# Cut points

For any fixed  $t > 0$ , a.s., the 2-dimensional Brownian path contains a closed loop around  $B_t$  in every interval  $(t, t + \varepsilon)$

Almost surely, there exist  $t \in (0, 1)$  such that  $B([0, t)) \cap B((t, 1]) = \emptyset$



# Intersection properties

$$(d = 1) \quad a.s., \forall t \exists s \neq t \quad B_s = B_t$$

$$(d = 2) \quad \forall t a.s. \forall s \neq t \quad B_s \neq B_t$$

$$a.s., \exists x \in R^2 \text{ Card}(B^{-1}(x)) = \infty$$

$$(d = 3) \quad a.s., \exists x \in R^3 \text{ Card}(B^{-1}(x)) = 2$$

$$a.s., \forall x \in R^3 \text{ Card}(B^{-1}(x)) \leq 2$$

$$(d = 4) \quad a.s., \forall x \in R^4 \text{ Card}(B^{-1}(x)) \leq 1$$

# Intersections of random sets

$$\dim(A) + \dim(B) > d$$



$$A \cap B \neq \emptyset$$

The dimension of Brownian trace is 2  
in every dimension.

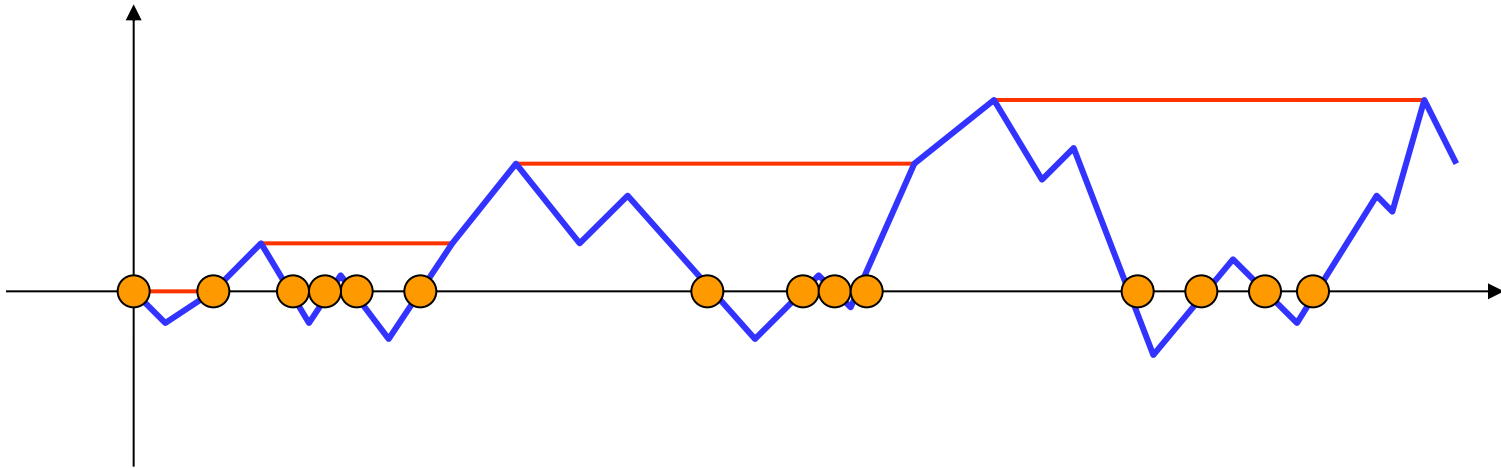
# Invariance principle

- (i) Random walk converges to Brownian motion (Donsker (1951))
- (ii) Reflected random walk converges to reflected Brownian motion (Stroock and Varadhan (1971) -  $C^2$  domains, B and Chen (2007) – uniform domains, not all domains)
- (iii) Self-avoiding random walk in 2 dimensions converges to SLE (200?) (open problem)

# Local time

$$L_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{-\varepsilon < B_s < \varepsilon\}} ds$$

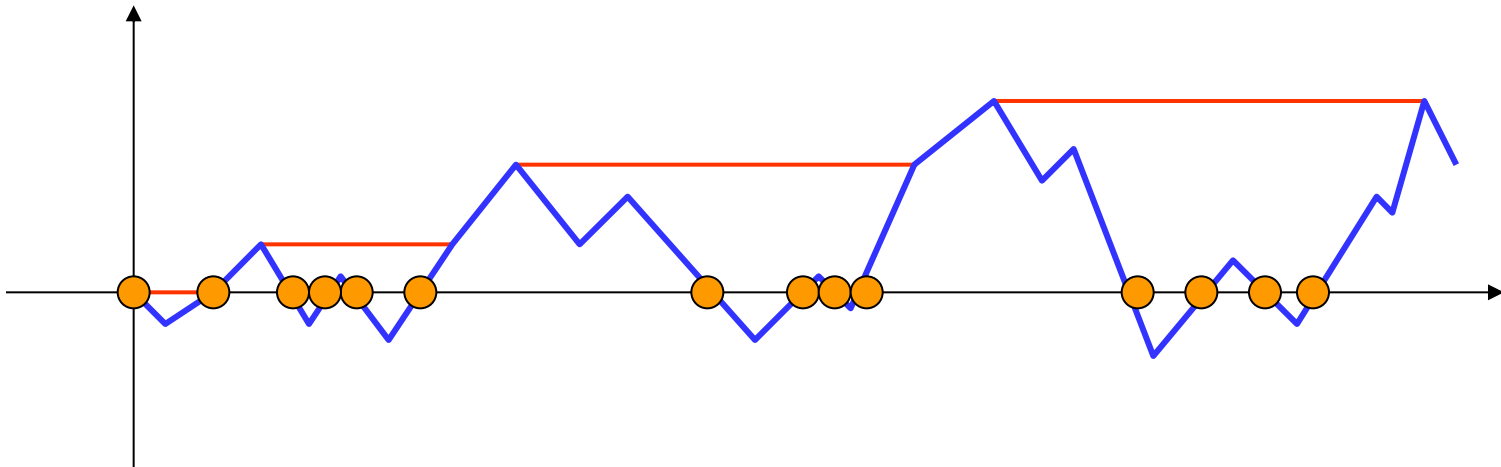
$$M_t = \sup_{s \leq t} B_s$$



## Local time (2)

$$\{L_t, 0 \leq t \leq 1\} \stackrel{D}{=} \{M_t, 0 \leq t \leq 1\}$$

$$\{M_t - B_t, 0 \leq t \leq 1\} \stackrel{D}{=} \{|B|_t, 0 \leq t \leq 1\}$$



## Local time (3)

$$\sigma_t = \inf_{s>0} \{L_s \geq t\}$$

Inverse local time is a stable process with index  $\frac{1}{2}$ .

# References

- R. Bass *Probabilistic Techniques in Analysis*, Springer, 1995
- F. Knight *Essentials of Brownian Motion and Diffusion*, AMS, 1981
- I. Karatzas and S. Shreve *Brownian Motion and Stochastic Calculus*, Springer, 1988